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SEASONAL QUASI-VECTOR AUTOREGRESSIVE MODELS FOR MACROECONOMIC DATA

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Abstract

We introduce the Seasonal-QVAR (quasi-vector autoregressive) model for world crude oil production and global real economic activity that identifies the hidden seasonality not found in linear VAR and VARMA models. World crude oil production has an annual seasonality component, and global real economic activity as measured by ocean freight rates has a six-month seasonality component. Seasonal-QVAR is a dynamic conditional score (DCS) model for the multivariate t distribution. Seasonal-VARMA and Seasonal-VAR are special cases of Seasonal-QVAR, this latter being superior to the two former models and also superior to the basic structural model with local level and stochastic seasonality components.

Keywords: Dynamic conditional score (DCS) models; score-driven stochastic seasonality; nonlinear multivariate dynamic location models; basic structural model; vector autoregressive (VAR) model; vector autoregressive moving average (VARMA) model; crude oil production

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I. Introduction

We introduce the new Seasonal-QVAR (quasi-vector autoregressive) model to be used for time series data from world crude oil production and global real economic activity. The use of the world crude oil production and global real economic activity variables is motivated by several works from the body of literature that study the question of how changes in supply and demand in the world crude oil market are related to economic growth (i.e., Blanchard, 2002; Barsky & Kilian, 2002, 2004; Hamilton, 2003; Kilian, 2008, 2009). World crude oil production has a significant annual seasonality component: During the summer months supply exceeds demand (i.e., relatively high crude oil production), while during the winter months demand exceeds supply (i.e., relatively low crude oil production) (Ye, Zyren & Shore, 2006). The global real economic activity variable used in this paper is from Kilian (2009), who uses dry cargo single voyage ocean freight rates to develop a global real economic activity index. The work of Kilian (2009) motivates the use of the freight rate-based global real economic activity index, as an alternative to the world industrial production variable. Ocean freight rates have a significant seasonality component, with a period of approximately six months (Raunglerdpanyagul, 1985).

The Gaussian unobserved components model with local level and stochastic seasonality, which is also named as the basic structural model (Harvey, 1989), is a widely-used model for macroeconomic and financial time series data. In the present paper, we suggest an alternative model that also includes a nonlinear stochastic seasonality component. The suggested model is an extension of the QVAR model that is also named as the 'dynamic conditional score model for the multivariate t distribution' (Harvey, 2013, chapter 7). Motivated by Harvey (2013, chapter 3.6), where nonlinear score-driven stochastic seasonality models are suggested (see also the related works of Harvey & Luati, 2014; Caivano, Harvey & Luati, 2016; Ayala & Blazsek, 2017; Blazsek & Hernández, 2017), we extend QVAR by adding a multivariate nonlinear score-driven stochastic seasonality component. We denote the new model as Seasonal-QVAR.

Seasonal-QVAR is a nonlinear multivariate dynamic conditional score (DCS) model, in which the conditional score of the log-likelihood (LL) function updates the vector of dependent variables

$y_t = (y_{1,t}, \dots, y_{K,t})'$. An advantage of QVAR, with respect to Gaussian multivariate models, is that the QVAR filter is robust to extreme values in the noise (Harvey, 2013).

For the QVAR(p) model, Blazsek, Escribano & Licht (2017) derive the nonlinear infinite vector moving average VMA(∞) representation of the local level, the corresponding impulse response function (IRF), and the conditions of consistency and asymptotic normality of the maximum likelihood (ML) estimator. Blazsek, Escribano & Licht (2017) find that the statistical performance of the QVAR model is superior to that of the VARMA (vector autoregressive moving average) and VAR models. For Seasonal-QVAR, we extend the results of Blazsek, Escribano & Licht (2017) to derive the VMA(∞) representation of the local level, the IRF, and the conditions of consistency and asymptotic normality of the ML estimator.

We find that Seasonal-QVAR effectively disentangles the local level and the stochastic seasonality components, and that it fits to the dataset better than the basic structural model. We present that Seasonal-VARMA and Seasonal-VAR are special cases of Seasonal-QVAR. Thus, our results can also be related to the VAR and VARMA literature (Sims, 1980, 1986; Sims, Goldfeld & Sachs, 1982; Bernanke, 1986; Blanchard & Watson, 1986; Tiao & Tsay, 1989; Stock & Watson, 2001). We find that Seasonal-QVAR is superior to both Seasonal-VARMA and Seasonal-VAR.

The remainder of this paper is organized as follows. Section II presents the econometric models. Section III describes the dataset and summarizes empirical results. Section IV concludes.

II. Econometric Models

A. Seasonal-QVAR

For y_t ($K \times 1$) with $t = 1, \dots, T$, the Seasonal-QVAR model is $y_t = c + \mu_t + s_t + v_t$, where c ($K \times 1$) includes constant parameters, μ_t ($K \times 1$) is the dynamic local level component, s_t ($K \times 1$) is the dynamic seasonality component, and v_t ($K \times 1$) is the reduced-form error term. Components μ_t and s_t are observable, conditional on the past information $(y_1, \dots, y_{t-1})$.

We formulate the reduced-form error term v_t as multivariate i.i.d. $v_t \sim t_K(0, \Sigma_v, \nu)$, where $\Sigma_v = \Omega_v^{-1}(\Omega_v^{-1})'$ is positive definite and $\nu > 2$ is the degrees of freedom parameter (hence, the variance of v_t is finite). As a consequence, $E(v_t) = 0$ and $\text{Var}(v_t) = \Sigma_v \times \nu/(\nu - 2)$. We also introduce the multivariate i.i.d. structural-form error term $\epsilon_t = [\nu/(\nu-2)]^{-1/2}\Omega_v \times v_t$, for which $E(\epsilon_t) = 0$ and $\text{Var}(\epsilon_t) = I_K$. As a consequence, the log of the conditional density of y_t is

$$\ln f(y_t|y_1,\ldots,y_{t-1}) = \ln\Gamma\left(\frac{\nu+K}{2}\right) - \ln\Gamma\left(\frac{\nu}{2}\right) - \frac{K}{2}\ln(\pi\nu) \tag{1}$$

$$-\frac{1}{2}\ln|\Sigma_v| - \frac{\nu+K}{2}\ln\left(1+\frac{v_t'\Sigma_v^{-1}v_t}{\nu}\right)$$

and the conditional score with respect to μ_t is

$$\frac{\partial \ln f(y_t|y_1,\ldots,y_{t-1})}{\partial\mu_t} = \frac{\nu+K}{\nu}\Sigma_v^{-1}\times\left(1+\frac{v_t'\Sigma_v^{-1}v_t}{\nu}\right)^{-1} v_t = \frac{\nu+K}{\nu}\Sigma_v^{-1}\times u_t \tag{2}$$

where u_t ($K \times 1$) is the scaled conditional score that updates the local level and the stochastic seasonality components. Harvey (2013, p. 211) shows that u_t is multivariate i.i.d. with zero mean and finite variance.

We formulate the local level component as $\mu_t = \Phi\mu_{t-1} + \Psi u_{t-1}$, where Φ ($K \times K$) and Ψ ($K \times K$) are time-constant parameter matrices, and μ_t is updated by the first lag of the scaled conditional score u_{t-1}. We initialize μ_t by using the unconditional mean $\mu_1 = E(\mu_1) = 0_{K\times 1}$. Alternatives to this initialization may also be used. For example, the elements of μ_1 may be estimated as additional parameters in the joint estimation of all parameters of Seasonal-QVAR (Harvey, 2013, p. 76). We use $\mu_1 = 0_{K\times 1}$, in order to reduce the number of estimated parameters.

We formulate each element of the seasonality component $s_t = (s_{1,t},\ldots,s_{K,t})'$ according to $s_{k,t} = D_t'\rho_{k,t}$ for each $k = 1,\ldots,K$, where $D_t = (D_{1,t},\ldots,D_{S,t})'$ is a vector of seasonal dummy variables and $\rho_{k,t} = (\rho_{k,1,t},\ldots,\rho_{k,S,t})'$ is a vector of dynamic seasonality parameters (S denotes the known period of the seasonality). Variable $\rho_{k,t}$ is updated according to $\rho_{k,t} = \rho_{k,t-1} + \gamma_{k,t}u_{k,t-1}$, where $\gamma_{k,t} = (\gamma_{k,1,t},\ldots,\gamma_{k,S,t})'$ is a dynamic scaling parameter and $u_{k,t-1}$ is the k-th element of u_{t-1}. It is noteworthy that $\partial\ln f(y_t|y_1,\ldots,y_{t-1})/\partial\mu_t = \partial\ln f(y_t|y_1,\ldots,y_{t-1})/\partial s_t$ from equation (1), hence, the same updating term u_{t-1} is used for μ_t and s_t (Harvey, 2013).

Furthermore, each element of $\gamma_{k,t}$ is given by $\gamma_{k,j,t} = \gamma_{k,j}$ for $D_{j,t} = 1$ and $\gamma_{k,j,t} = -\gamma_{k,j}/(S-1)$ for $D_{j,t} = 0$, where $\gamma_{k,j}$ with $j = 1, \dots, S$ are seasonality parameters to be estimated (Harvey, 2013, p. 80). This parameterization ensures that $\sum_{j=1}^{S} \gamma_{k,j,t} = 0$. As a consequence, $s_{k,t}$ is centered at zero for each dependent variable (we demonstrate this in the appendix). Thus, $s_{k,t}$ is a high-pass filter (trend-reducing filter) that compensates the unit root in $\rho_{k,t}$ (Baxter & King, 1999), since $y_{k,t}$ is $I(0)$.

We initialize $\rho_{k,t}$ by using a first-step nonlinear least squares (NLS) estimation procedure, in which we regress $y_{k,t}$ on the seasonal dummy variables, under the restriction that the sum of all parameters is zero (Harvey, 2013, p. 80). This parameterization ensures for $t = 1$ that $\sum_{j=1}^{S} \rho_{k,j,1} = 0$. As a consequence, $s_{k,1}$ is centered at zero for each dependent variable (see the appendix). Alternatives to this initialization may also be considered (see Harvey, 2013). However, the NLS procedure used for initialization, turns out to be very useful for the effective estimation of Seasonal-QVAR. Due to this initialization of seasonality, the Seasonal-QVAR specification used in this paper is able to disentangle the dynamic interaction effects measured by μ_t from the stochastic seasonality effects measured by s_t.

It is noteworthy that, in Seasonal-QVAR, the local level component μ_t measures all dynamic interaction effects among y_t. Therefore, we focus on the following structural-form nonlinear VMA(∞) representation of the local level component (Blazsek, Escribano & Licht, 2017):

$$\mu_t = \sum_{j=0}^{\infty} \Phi^j \Psi [(\nu - 2)\nu]^{1/2} \Omega_v^{-1} \frac{\epsilon_{t-1-j}}{\nu - 2 + \epsilon'_{t-1-j}\epsilon_{t-1-j}} \tag{3}$$

Let C_1 denote the maximum modulus of eigenvalues of Φ. The series in equation (3) is convergent if $C_1 < 1$. $\text{IRF}_{j,t} = \partial \mu_{t+j}/\partial \epsilon_t$ is given by (Blazsek, Escribano & Licht, 2017):

$$\text{IRF}_{j,t} = \Phi^j \Psi [(\nu - 2)\nu]^{1/2} \Omega_v^{-1} Q_{t-1-j} \quad \text{for} \quad j = 1, \dots, \infty \tag{4}$$

where

$$Q_t = \frac{\partial \frac{\epsilon_t}{\nu-2+\epsilon_t'\epsilon_t}}{\partial \epsilon_t} = \begin{bmatrix} q_{1,1,t} & \cdots & q_{1,K,t} \\ \cdots & \cdots & \cdots \\ q_{K,1,t} & \cdots & q_{K,K,t} \end{bmatrix} = \tag{5}$$

$$= \begin{bmatrix} \frac{\nu-2+\epsilon_t'\epsilon_t-2\epsilon_{1,t}^2}{(\nu-2+\epsilon_t'\epsilon_t)^2} & \frac{-2\epsilon_{1,t}\epsilon_{2,t}}{(\nu-2+\epsilon_t'\epsilon_t)^2} & \cdots & \frac{-2\epsilon_{1,t}\epsilon_{K,t}}{(\nu-2+\epsilon_t'\epsilon_t)^2} \\ \frac{-2\epsilon_{2,t}\epsilon_{1,t}}{(\nu-2+\epsilon_t'\epsilon_t)^2} & \frac{\nu-2+\epsilon_t'\epsilon_t-2\epsilon_{2,t}^2}{(\nu-2+\epsilon_t'\epsilon_t)^2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{-2\epsilon_{K,t}\epsilon_{1,t}}{(\nu-2+\epsilon_t'\epsilon_t)^2} & \cdots & \cdots & \frac{\nu-2+\epsilon_t'\epsilon_t-2\epsilon_{K,t}^2}{(\nu-2+\epsilon_t'\epsilon_t)^2} \end{bmatrix}$$

As $\mathrm{IRF}_{j,t}$ depends on t, we use its unconditional mean (Blazsek, Escribano & Licht, 2017):

$$\mathrm{IRF}_j = E(\mathrm{IRF}_{j,t}) = \Phi^j \Psi[(\nu-2)\nu]^{1/2}\Omega_v^{-1}E(Q_{t-1-j}) \quad \text{for} \quad j = 1, 2, \ldots, \infty \tag{6}$$

If all elements of Q_t are covariance stationary, then $E(Q_{t-1-j})$ can be estimated by using the sample average (Hamilton, 1994, chapter 7.2). We test the covariance stationarity of Q_t by using the augmented Dickey–Fuller (1979) test (ADF) with constant.

B. Statistical Inference of Seasonal-QVAR

We estimate Seasonal-QVAR by using the maximum likelihood (ML) method (Davidson & MacKinnon, 2003). The ML estimator of parameters is

$$\hat{\Theta}_{\mathrm{ML}} = \arg\max_{\Theta} \mathrm{LL}(y_1, \ldots, y_T) = \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t | y_1, \ldots, y_{t-1}) \tag{7}$$

where Θ denotes the vector of time-constant parameters. We use the numerically estimated inverse information matrix (Harvey, 2013, p. 51) to obtain ML standard errors.

We use results from Harvey (2013, chapters 2.3, 2.4 and 3.3) for the conditions under which ML is consistent and asymptotically normal (see also Blazsek, Escribano & Licht, 2017). First, condition 1 is $C_1 < 1$, hence, μ_t is covariance stationary. Second, we use condition 2 of Harvey

(2013, p. 35). Condition 2 holds if $E[u_{j,t}^{2-i}(\partial u_{k,t}/\partial \mu_{l,t})^i] < \infty$, where $i = 0, 1, 2$ and $j, k, l = 1, \ldots, K$ (we test condition 2 by using ADF). Third, for condition 3, we use Harvey (2013, p. 49, Theorem 5). We consider the representative element $\Psi_{i,j}$ from the matrix Ψ. From the dynamic local level equation $\mu_t = \Phi\mu_{t-1} + \Psi u_{t-1}$, we express

$$\frac{\partial \mu_t}{\partial \Psi_{i,j}} = \Phi \frac{\partial \mu_{t-1}}{\partial \Psi_{i,j}} + \Psi \frac{\partial u_{t-1}}{\partial \Psi_{i,j}} + W_{i,j} u_{t-1} \tag{8}$$

for all $t = 1, \ldots, T$, where the element (i, j) of the matrix $W_{i,j}$ $(K \times K)$ is one and the rest of the elements of $W_{i,j}$ are zero. By using the chain rule, we can also write equation (8) as

$$\frac{\partial \mu_t}{\partial \Psi_{i,j}} = \left(\Phi + \Psi \frac{\partial u_{t-1}}{\partial \mu'_{t-1}}\right) \frac{\partial \mu_{t-1}}{\partial \Psi_{i,j}} + W_{i,j} u_{t-1} = X_t \frac{\partial \mu_{t-1}}{\partial \Psi_{i,j}} + W_{i,j} u_{t-1} \tag{9}$$

Condition 3 is that all eigenvalues of $E(X_t)$ are within the unit circle, where the finiteness of all elements of $E(X_t)$ follows from condition 2. We denote the maximum modulus of eigenvalues of $E(X_t)$ by using C_3. If each element of X_t is covariance stationary, then $E(X_t)$ can be estimated by using the sample average (we test covariance stationarity by using ADF). Fourth, for condition 4, we consider that the information matrix depends on:

$$\frac{\partial \mu_t}{\partial \Psi_{i,j}} \frac{\partial \mu'_t}{\partial \Psi_{k,l}} = X_t \frac{\partial \mu_{t-1}}{\partial \Psi_{i,j}} \frac{\partial \mu'_{t-1}}{\partial \Psi_{k,l}} X'_t + X_t \frac{\partial \mu_{t-1}}{\partial \Psi_{i,j}} W'_{i,j} u_{t-1} + u'_{t-1} W_{k,l} \frac{\partial \mu'_{t-1}}{\partial \Psi_{k,l}} X'_t + W_{i,j} u_{t-1} u'_{t-1} W'_{k,l} \tag{10}$$

that we can also write as

$$\mathrm{vec}\left(\frac{\partial \mu_t}{\partial \Psi_{i,j}} \frac{\partial \mu'_t}{\partial \Psi_{k,l}}\right) = (X_t \otimes X_t)\mathrm{vec}\left(\frac{\partial \mu_{t-1}}{\partial \Psi_{i,j}} \frac{\partial \mu'_{t-1}}{\partial \Psi_{k,l}}\right) + \tag{11}$$

$$+\mathrm{vec}\left(X_t \frac{\partial \mu_{t-1}}{\partial \Psi_{i,j}} W'_{i,j} u_{t-1}\right) + \mathrm{vec}\left(u'_{t-1} W_{k,l} \frac{\partial \mu'_{t-1}}{\partial \Psi_{k,l}} X'_t\right) + \mathrm{vec}\left(W_{i,j} u_{t-1} u'_{t-1} W'_{k,l}\right)$$

where $\otimes$ is the Kronecker product and vec(x) indicates that the columns of the matrix are being stacked one upon the other. Condition 4 is that all eigenvalues of $E(X_t \otimes X_t)$ are within the unit circle, where the finiteness of all elements of $E(X_t \otimes X_t)$ follows from condition 2. We denote the maximum modulus of eigenvalues of $E(X_t \otimes X_t)$ by using C_4. If each element of $X_t \otimes X_t$ is covariance stationary, then $E(X_t \otimes X_t)$ can be estimated by using the sample average (we test the covariance stationarity of $X_t \otimes X_t$ by using ADF). It is noteworthy that for the computation of $X_t = \Phi + \Psi(\partial u_{t-1}/\partial \mu'_{t-1})$ in conditions 3 and 4, we use the formula for $\partial u_t/\partial \mu'_t$ $(K \times K)$. As aforementioned, the score function is given by

$$u_t = \left(1 + \frac{v'_t \Sigma_v^{-1} v_t}{\nu}\right)^{-1} v_t = \frac{\nu(y_t - c - \mu_t - s_t)}{\nu + (y_t - c - \mu_t - s_t)' \Sigma_v^{-1}(y_t - c - \mu_t - s_t)} \tag{12}$$

and the formula of $\partial u_t/\partial \mu'_t$ can be obtained by using standard matrix calculus.

Finally, for the seasonality component s_t, the conditions of ML are satisfied if the parameters of $\rho_{k,t-1}$ are set to the identity matrix (i.e., multivariate random walk), as in the equation $\rho_{k,t} = \rho_{k,t-1} + \gamma_{k,t} u_{k,t-1}$. If the unit root is imposed rather than estimated in DCS models of trend or seasonality, then standard asymptotics of ML do apply (Harvey, 2013, p. 49).

C. Seasonal-VARMA and Seasonal-VAR

Seasonal-QVAR can be related to Seasonal-VARMA and Seasonal-VAR. With respect to VAR, we refer to the important works of Sims (1980, 1986), Sims, Goldfeld & Sachs (1982), Bernanke (1986), Blanchard & Watson (1986), and Stock & Watson (2001). With respect to VARMA, we refer to the seminal paper of Tiao & Tsay (1989). We also refer to the related textbooks of Hamilton (1994, chapters 10 to 12) and Lütkepohl (2005).

If $\nu \to \infty$, then $v_t \to_d N_K(0, \Sigma_v)$ and $u_t \to_p v_t$. For this limiting case, from equations $y_t = c + \mu_t + s_t + v_t$ and $\mu_t = \Phi\mu_{t-1} + \Psi u_{t-1}$, we obtain the Gaussian version of Seasonal-QVAR:

$$y_t = (I_K - \Phi)c + \Phi y_{t-1} + (I_K - \Phi L)s_t + (\Psi - \Phi)v_{t-1} + v_t \tag{13}$$

where L denotes the lag operator. Thus, Seasonal-QVAR becomes Seasonal-VARMA, in which the error term v_t has a multivariate i.i.d. Gaussian distribution and both local level μ_t and stochastic seasonality s_t are updated by v_{t-1}. Seasonal-VARMA is a special case of Seasonal-

QVAR for large ν. Furthermore, Seasonal-VAR is another special (and improbable) case of Seasonal-QVAR for large ν and $\Psi = \Phi$. These suggest that, with respect to model fit, Seasonal-QVAR should be superior to Seasonal-VARMA and Seasonal-VAR.

For Seasonal-VARMA and Seasonal-VAR, the local level μ_t measures all dynamic interaction effects among y_t. Therefore, we focus on the VMA(∞) representation $\mu_t = \sum_{j=0}^{\infty} \Phi^j \Psi \Omega_v^{-1} \epsilon_{t-j}$, which is obtained by using the decomposition $\Sigma_v = \Omega_v^{-1}(\Omega_v^{-1})'$. Let C_1 denote the maximum modulus of eigenvalues of Φ. If $C_1 < 1$, then the series in the VMA(∞) representation is finite. For Seasonal-VARMA, $\mathrm{IRF}_{j,t} = \partial \mu_{t+j} / \partial \epsilon_t = \Phi^j \Psi \Omega_v^{-1}$ for $j = 1, \ldots, \infty$. For Seasonal-VAR, $\mathrm{IRF}_{j,t} = \partial \mu_{t+j} / \partial \epsilon_t = \Phi^{j+1} \Omega_v^{-1}$ for $j = 1, \ldots, \infty$.

We estimate Seasonal-VARMA and Seasonal-VAR by using the quasi-ML (QML) method (Gouriéroux, Monfort & Trognon, 1984), for which we use $v_t \sim N_K(0, \Sigma_v)$ as a pseudo distribution. We denote the maximum moduli of eigenvalues of Φ and $\Psi - \Phi$ with C_1 and C_2, respectively. For Seasonal-VARMA, $C_1 < 1$ and $C_2 < 1$ ensure that QML is consistent and asymptotically normal. For Seasonal-VAR, $C_1 < 1$ is required for those asymptotic properties.

D. Basic Structural Model

The basic structural model is $y_t = c + \mu_t + s_t + v_t$, where c ($K \times 1$) includes constant parameters, μ_t ($K \times 1$) is the local level component, s_t ($K \times 1$) is the stochastic seasonality component, and $v_t \sim N_K(0_{K\times 1}, \Sigma_v)$ is a multivariate i.i.d. Gaussian error term. The positive definite covariance matrix is decomposed as $\Sigma_v = \Omega_v^{-1}(\Omega_v^{-1})'$. We formulate the local level component as $\mu_t = \Phi \mu_{t-1} + \eta_t$, where Φ ($K \times K$) is a constant parameter matrix, and $\eta_t \sim N_K(0_{K\times 1}, \Sigma_\eta)$ is the multivariate i.i.d. reduced-form error term. The covariance matrix is decomposed as $\Sigma_\eta = \Omega_\eta^{-1}(\Omega_\eta^{-1})'$, and we introduce the multivariate i.i.d. structural-form error term $\epsilon_t = \Omega_\eta \eta_t$. We initialize μ_t in the same way as for Seasonal-QVAR. We formulate each element of the seasonality component $s_t = (s_{1,t}, \ldots, s_{K,t})'$ according to the product $s_{k,t} = D_t' \rho_{k,t}$ for each $k = 1, \ldots, K$, where D_t ($S \times 1$) is a vector of seasonal dummy variables and $\rho_{k,t}$ ($S \times 1$) is a vector of dynamic seasonality parameters. Variable ρ_{kt} is updated according to the dynamic equation $\rho_{k,t} = \rho_{k,t-1} + \xi_{k,t}$, where $\xi_{k,t} \sim N_S(0, \Sigma_{\xi,k})$ is a multivariate i.i.d. Gaussian error term.

We ensure that $s_{k,t}$ is centered at zero by using the specification $\Sigma_{\xi,k} = \sigma_{\xi,k}(I_S - i_S i_S'/S)$, where $\sigma_{\xi,k}$ is a positive parameter and i_S denotes a $S \times 1$ vector of ones (Harvey, 2013, p. 79). We initialize $\rho_{k,t}$ by using the same first-step NLS procedure that is used for Seasonal-QVAR.

For the basic structural model estimated in this paper, the local level component μ_t measures all dynamic interaction effects among y_t. Therefore, we focus on the following VMA(∞) representation: $\mu_t = \sum_{j=0}^{\infty} \Phi^j \Omega_\eta^{-1} \epsilon_{t-j}$. Let C_1 denote the maximum modulus of eigenvalues of Φ. If $C_1 < 1$, then $\text{IRF}_{j,t} = \partial\mu_{t+j}/\partial\epsilon_t = \Phi^j \Omega_\eta^{-1}$ for $j = 1, \ldots, \infty$. We estimate the basic structural model by using the ML method, for which the likelihood function is computed by using the Kalman filter technique (Kalman, 1960; Harvey, 1989).

III. Data and Empirical Results

A. Macroeconomic Dataset

We use monthly time series data from change in world crude oil production $y_{1,t}$, and change in a business cycle index measuring global real economic activity $y_{2,t}$, for the period 1973 to 2007 (source: Kilian & Lütkepohl, 2017; http://www-personal.umich.edu/~lkilian/figure12_7.zip). The use of those variables is motivated by several works from the body of literature that study the question of how changes in supply and demand in the world crude oil market are related to economic growth (i.e., Blanchard, 2002; Barsky & Kilian, 2002, 2004; Hamilton, 2003; Kilian, 2008, 2009; Kilian & Lütkepohl, 2017). World crude oil production has a significant annual seasonality component: During the summer months supply exceeds demand (i.e., relatively high crude oil production), while during the winter months demand exceeds supply (i.e., relatively low crude oil production) (Ye, Zyren & Shore, 2006). As aforementioned, the global real economic activity variable of this paper is developed by Kilian (2009), who uses dry cargo single voyage ocean freight rates to measure global real economic activity. The work of Kilian (2009) motivates the use of the ocean freight rate-based global real economic activity index, as an alternative to the world industrial production variable. Ocean freight rates have a significant seasonality component, with a period of about six months (Raunglerdpanyagul, 1985).

In our application, $y_t = (y_{1,t}, y_{2,t})'$ (thus, for the multivariate models of this paper $K = 2$), and we use the seasonal dummies $D_t = (D_{\text{Jan},t}, \ldots, D_{\text{Dec},t})'$ (thus, the period of the annual seasonality used in this paper is $S = 12$). For $y_{1,t}$ and $y_{2,t}$, descriptive statistics and ADF with constant test results are reported in table 1. Robust ordinary least squares (OLS) estimates (Newey & West, 1987) of a linear regression of $y_{1,t}$ and $y_{2,t}$ on monthly dummies are also reported in table 1, which suggest that both variables may have seasonality components.

It is noteworthy that, motivated by the work of Kilian & Lütkepohl (2017, chapter 12.13.1), all multivariate dynamic models of the dataset used in this paper can be identified recursively. Therefore, Σ_v for Seasonal-QVAR, Seasonal-VARMA and Seasonal-VAR is decomposed according to the Cholesky decomposition. Similarly, Σ_v and Σ_η for the basic structural model are also decomposed according to the Cholesky decomposition.

[APPROXIMATE LOCATION OF TABLE 1]

B. Disentanglement of Local Level and Seasonality

In this section, we assume that the data generating process (DGP) includes a seasonality component that we would like to disentangle from the local level component. For the extreme case where seasonality is not specified in QVAR (i.e., $y_t = c + \mu_t + v_t$ and $\mu_t = \Phi\mu_{t-1} + \Psi u_{t-1}$), we demonstrate that dynamic seasonality effects will appear in the local level component. This is indicated by the fact that seasonality effects will be observed in the IRF of μ_t. Thus, for a given Seasonal-QVAR specification, the IRF is a useful tool that analyzes the effectiveness of information disentanglement for μ_t and s_t. If the IRF does not indicate seasonality dynamics, then s_t will capture all seasonality effects and the disentanglement of μ_t and s_t is effective.

We present the parameter estimates and model diagnostics of QVAR (i.e., $y_t = c + \mu_t + v_t$) in table 2. This table indicates that we were not able to estimate QVAR for the case where all elements of Ψ are estimated ('Ψ full' in table 2), as the ML estimator did not converge to an optimum. This suggests that QVAR without the seasonality component is a misspecified model. We are able to estimate QVAR for the restricted case, where Ψ is a diagonal matrix ('Ψ diagonal' in table 2) (for that specification, ML is supported by C_1 to C_4, and the MDS test does not reject the specifications of ϵ_t and u_t; table 2). We present the IRF of μ_t for QVAR with Ψ diagonal in figure 1, where the IRF estimates indicate dynamic seasonality effects in μ_t. These preliminary findings suggest that, for Seasonal-QVAR, the effectiveness of disentanglement of the local level and the stochastic seasonality components can be analyzed by using the IRF tool.

We also demonstrate in this section, for the extreme case where the seasonality component is not included in VARMA or VAR (i.e., $y_t = c + \mu_t + v_t$ and $\mu_t = \Phi \mu_{t-1} + \Psi v_{t-1}$, with the possibility of $\Psi = \Phi$ for VAR), that seasonality effects will not appear in the local level component of the model (i.e., seasonality effects will not be observed in the IRF of μ_t). We present the parameter estimates and diagnostics of VARMA and VAR in table 2 and the corresponding IRF in figure 1. We estimated VARMA with the Ψ full and also with the Ψ diagonal specifications (C_1 and C_2 support the asymptotic properties of QML for both VARMA and VAR, and the MDS test does not reject the specification of ϵ_t; table 2). We find that the statistical performance of VARMA and VAR is inferior to that of QVAR, according to the LL, Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan–Quinn criterion (HQC) metrics (Davidson & MacKinnon, 2003) (table 2). More importantly, although the DGP includes a seasonality component, the related seasonality effects do not appear in the IRF of VARMA and VAR (figure 1). As a consequence, for Seasonal-VARMA and Seasonal-VAR, the effectiveness of the disentanglement of μ_t and s_t cannot be analyzed by using the IRF tool.

[APPROXIMATE LOCATION OF TABLE 2 AND FIGURE 1]

C. Empirical Results

We present the parameter estimates and diagnostics for Seasonal-QVAR and the basic structural model in table 3, and the time series components of those models in figures 2 and 3, respectively. We present the parameter estimates and diagnostics for Seasonal-VARMA and Seasonal-VAR in table 4, and the time series components of those models in figures 4 and 5, respectively. We present the IRF of Seasonal-QVAR, the basic structural model, Seasonal-VARMA and Seasonal-VAR in figure 6. We present Seasonal-QVAR is robust to extreme values in the noise in figure 7.

The most important findings are the following. First, consistency and asymptotic normality of the estimates are supported by C_1, C_2, C_3 or C_4 for each model (tables 3 and 4). Second, we use the Escanciano–Lobato (2009) martingale difference sequence (MDS) test with optimal lag order for ϵ_t and u_t of Seasonal-QVAR (as suggested by Harvey, 2013) and v_t of the basic structural model. We find that MDS is never rejected (table 3). We use the same MDS test for ϵ_t of Seasonal-VARMA and Seasonal-VAR. We find that MDS is never rejected (table 4).

Third, we compare statistical performances by using the LL, AIC, BIC and HQC metrics. LL and AIC indicate some improvement in the model performance of Seasonal-QVAR, with respect to the basic structural model (table 3). However, BIC and HQC indicate a superior performance for the basic structural model, which is due to the fact that the number of parameters of the basic structural model is much lower than that of Seasonal-QVAR (table 3). Thus, the basic structural model might be more parsimonious than Seasonal-QVAR. With respect to Seasonal-VARMA and Seasonal-VAR, all metrics indicate that those models are inferior to both Seasonal-QVAR and the basic structural model (tables 3 and 4).

It is noteworthy that LL of QVAR is superior to that of Seasonal-QVAR (see tables 2 and 3). This result might seem surprising, as one might believe that QVAR is a nested alternative to Seasonal-QVAR. In fact, QVAR is not a special case of the Seasonal-QVAR model estimated in this paper, since we use a specific restriction for Seasonal-QVAR about the initial value of $\rho_{k,t}$ that is estimated in the first-step NLS procedure. QVAR is estimated without such restriction. Nevertheless, as aforementioned, the restriction about the initial value of $\rho_{k,t}$ is important for the disentanglement of μ_t and s_t in Seasonal-QVAR.

Fourth, $s_{1,t}$ and $s_{2,t}$ in figures 2 to 5 indicate significant seasonality effects with dynamic amplitude for all models, which supports the use of stochastic seasonality. It is noteworthy that the local level estimates $\mu_{1,t}$ and $\mu_{2,t}$ are relatively homogeneous for Seasonal-QVAR, as compared to the Gaussian alternatives (figures 2 to 5). This suggests that extreme observations tend to appear in the error term of Seasonal-QVAR, while extreme observations sometimes appear in the local level components of the basic structural model, Seasonal-VARMA and Seasonal-VAR.

Fifth, we study interaction effects between world crude oil production and global real economic activity by using the IRF (figures 6). The results are similar for Seasonal-QVAR, the basic structural model and Seasonal-VARMA, and they show that seasonality does not appear in the IRF of μ_t. This point is important for Seasonal-QVAR, since it indicates that the specification applied in this paper effectively disentangles μ_t and s_t. For Seasonal-VAR, we find the opposite dynamic interaction effects between variables for two panels in figure 6, as compared to alternative models. This indicates that Seasonal-VAR is a misspecified model.

Sixth, for Seasonal-QVAR, the conditional score $u_t = (u_{1t}, u_{2t})'$ discounts extreme values from the structural-form error term $\epsilon_t = (\epsilon_{1t}, \epsilon_{2t})'$. In figure 7, we present each element of the updating vector $u_t = (u_{1t}, u_{2t})'$ of Seasonal-QVAR, as a function of ϵ_{1t} and ϵ_{2t}. This figure indicates that both elements of the conditional score converge to finite values, when $|\epsilon_{1t}|$ or $|\epsilon_{2t}|$ go to infinity. Thus, for Seasonal-QVAR, the conditional score discounts the influence of extreme values in the noise. To compare the discounting property of the updating terms for Seasonal-QVAR and Seasonal-VARMA, in figure 7, we also present each element of the updating vector $v_t = (v_{1t}, v_{2t})'$ of Seasonal-VARMA, as a function of ϵ_{1t} and ϵ_{2t}. This figure indicates that v_{1t} or v_{2t} converge to infinity, when $|\epsilon_{1t}|$ and $|\epsilon_{2t}|$ go to infinity. Thus, for Seasonal-VARMA, the updating terms v_{1t} and v_{2t} do not discount the influence of extreme values in the noise.

Finally, with respect to the identification of μ_t and s_t in Seasonal-QVAR, two points are noteworthy: (i) Initially, we undertook the separate estimation of s_t as specified in Seasonal-QVAR, with the objective of deseasonalizing y_t in a first step. We used the residuals estimated for s_t as dependent variables of QVAR without the seasonality component, in a second step. The IRF results of QVAR showed that deseasonalization by using this two-step procedure was not effective, since the IRF of QVAR had seasonal dynamics. (ii) As aforementioned, for Seasonal-QVAR, the first-step NLS estimation of the initial values of $\rho_{k,t}$ that drives seasonality is very useful for the effective separation of μ_t and s_t. As an alternative, we also estimated Seasonal-QVAR by using zero initial values for all elements of $\rho_{k,t}$. The IRF of Seasonal-QVAR corresponding to this initialization showed seasonal dynamics, thus, the separation of μ_t and s_t was not effective.

[APPROXIMATE LOCATION OF TABLES 3-4 AND FIGURES 2-7]

IV. Conclusions

We have introduced the new Seasonal-QVAR model, to be used for time series data from world crude oil production and global real economic activity, that is able to identify the hidden seasonality not found in linear VAR and VARMA models. World crude oil production has a significant annual seasonality component, and global real economic activity has a significant seasonality component with a period of approximately six months. We have extended the recent QVAR model by adding a multivariate nonlinear score-driven stochastic seasonality component, and denoted the new model as Seasonal-QVAR. This model is an alternative to the basic structural model. For Seasonal-QVAR, we have presented the nonlinear infinite VMA representation of the local level component, the corresponding IRF, and the conditions of asymptotic properties of ML. We have found that Seasonal-QVAR effectively disentangles the local level and the stochastic seasonality components. We have presented that Seasonal-VARMA and Seasonal-VAR are special cases of Seasonal-QVAR, and we have demonstrated that the statistical performance of Seasonal-QVAR is superior to those alternatives. We have compared Seasonal-QVAR with the benchmark basic structural model. The results have suggested an improvement in the model fit of Seasonal-QVAR, with respect to the benchmark model.

Appendix

In this appendix, we show that the seasonality component $s_t = (s_{1,t}, \ldots, s_{K,t})'$ is centered at zero, and it is a high-pass filter that compensates the unit root in $\rho_{k,t}$, since $y_{k,t}$ is $I(0)$ or covariance stationary. Each element of the seasonality component is modeled as:

$$s_{k,t} = D_t'\rho_{k,t} = D_{1,t}\rho_{k,1,t} + D_{2,t}\rho_{k,2,t} + \ldots + D_{S,t}\rho_{k,S,t} \tag{A.1}$$

The vector of dynamic seasonal parameters is $\rho_{k,1}$ for $t = 1$ and $\rho_{k,t} = \rho_{k,t-1} + \gamma_{k,t}u_{k,t-1}$ for $t = 2, \ldots, T$. By recursive substitution, we obtain that $\rho_{k,t} = \rho_{k,1} + \gamma_{k,2}u_{k,1} + \ldots + \gamma_{k,t}u_{k,t-1}$, where $\rho_{k,1}$ is the vector of initial values of $\rho_{k,t}$. Then, each element of $\rho_{k,t}$ is given by

$$\rho_{k,j,t} = \rho_{k,j,1} + \gamma_{k,j,2} u_{k,1} + \ldots + \gamma_{k,j,t} u_{k,t-1} \tag{A.2}$$

for $j = 1, \ldots, S$. Substituting equation (A.2) into equation (A.1) we get:

$$\begin{aligned} s_{k,t} &= D_{1,t}(\rho_{k,1,1} + \gamma_{k,1,2} u_{k,1} + \ldots + \gamma_{k,1,t} u_{k,t-1}) + \\ &+ D_{2,t}(\rho_{k,2,1} + \gamma_{k,2,2} u_{k,1} + \ldots + \gamma_{k,2,t} u_{k,t-1}) + \\ &+ D_{3,t}(\rho_{k,3,1} + \gamma_{k,3,2} u_{k,1} + \ldots + \gamma_{k,3,t} u_{k,t-1}) + \\ &+ \vdots \\ &+ D_{S,t}(\rho_{k,S,1} + \gamma_{k,S,2} u_{k,1} + \ldots + \gamma_{k,S,t} u_{k,t-1}) \end{aligned} \tag{A.3}$$

In equation (A.3), the dummy variables select each one of the terms consecutively for each t. The selected value of $s_{k,t}$ is zero on average for consecutive $j = 1, \ldots, S$ time periods, because the average of each term within the parentheses of equation (A.3) is zero. To see this, consider first the NLS procedure used for the estimation of the initial values of $\rho_{k,t}$, which ensures that $\rho_{k,1,1} + \rho_{k,2,1} + \ldots + \rho_{k,S,1} = 0$. Thus, $(\rho_{k,1,1} + \rho_{k,2,1} + \ldots + \rho_{k,S,1})/S = 0$. In addition, for all terms where $\gamma_{k,j,t}$ appears, $\gamma_{k,1,t} + \gamma_{k,2,t} + \ldots + \gamma_{k,S,t} = 0$, because for $j = 1, \ldots, S$, we parameterize $\gamma_{k,j,t} = \gamma_{k,j}$ for $D_{j,t} = 1$ and $\gamma_{k,j,t} = -\gamma_{k,j}/(S-1)$ for $D_{j,t} = 0$. Thus, $(\gamma_{k,1,t} u_{k,t-1} + \gamma_{k,2,t} u_{k,t-1} + \ldots + \gamma_{k,S,t} u_{k,t-1})/S = 0$. Therefore, the average of $s_{k,t}$ is also zero for consecutive $j = 1, \ldots, S$ time periods, and the seasonality component $s_t = (s_{1,t}, \ldots, s_{K,t})'$ is centered at zero.

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TABLE 1.—DESCRIPTIVE STATISTICS

Panel A. Descriptive statistics	World crude oil production $y_{1,t}$	Global real economic activity $y_{2,t}$
Start date	March 1973	March 1973
End date	December 2007	December 2007
Sample size T	418	418
Minimum	-9.9073	-20.8529
Maximum	6.4986	17.0466
Mean	0.0719	0.0497
Standard deviation	1.7117	4.8134
Skewness	-1.5326	-0.2381
Kurtosis	8.1718	1.8608
ADF	-22.3144^{***}	-15.2541^{***}
Panel B. Seasonality effects	**World crude oil production y_{1t}**	**Global real economic activity y_{2t}**
δ_{Jan}	$-1.2007^{***}(0.4342)$	$-2.0955^{***}(0.768)$
δ_{Feb}	$0.3447(0.3137)$	$-1.0236(0.6247)$
δ_{Mar}	$0.0716(0.1882)$	$2.0131^{**}(0.8216)$
δ_{Apr}	$-0.2675(0.202)$	$-0.4418(0.6527)$
δ_{May}	$-0.1369(0.2217)$	$1.248(0.7759)$
δ_{Jun}	$0.1075(0.2525)$	$-2.9096^{***}(0.8132)$
δ_{Jul}	$0.7465^{***}(0.257)$	$-2.8145^{***}(0.7816)$
δ_{Aug}	$-0.213(0.2634)$	$0.2747(0.6942)$
δ_{Sep}	$0.5944^{*}(0.3122)$	$3.0974^{***}(0.6693)$
δ_{Oct}	$0.3188(0.3721)$	$2.8003^{***}(0.7255)$
δ_{Nov}	$0.3663^{*}(0.2105)$	$1.853^{***}(0.5436)$
δ_{Dec}	$0.1032(0.2071)$	$-1.4976^{*}(0.888)$

Robust standard errors are in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Table 2.—Parameter Estimates and Model Diagnostics (QVAR; VARMA; VAR)

	QVAR Ψ full	QVAR Ψ diagonal	VARMA Ψ full	VARMA Ψ diagonal	VAR
c_1	NA	0.1740***(0.0571)	0.0337(0.0261)	0.0247(0.0163)	0.0778(0.0780)
c_2	NA	0.0915(0.2618)	0.1491(0.2685)	0.0515(0.2587)	0.0409(0.0371)
$\Phi_{1,1}$	NA	−0.7282***(0.2123)	0.5557***(0.1164)	0.6461***(0.0830)	−0.0961**(0.0478)
$\Phi_{1,2}$	NA	0.0983*(0.0525)	−0.0191(0.0340)	0.0151*(0.0084)	0.0221(0.0162)
$\Phi_{2,1}$	NA	−2.0782(1.6415)	−1.4298*(0.7634)	−0.0363(0.1169)	−0.0029(0.1217)
$\Phi_{2,2}$	NA	0.4562***(0.1494)	0.2133(0.1520)	0.1214**(0.0555)	0.2831***(0.0445)
$\Psi_{1,1}$	NA	0.0593(0.0798)	−0.7243***(0.1021)	−0.8129***(0.0640)	NA
$\Psi_{1,2}$	NA	NA	0.0451(0.0460)	NA	NA
$\Psi_{2,1}$	NA	NA	1.4934*(0.7819)	NA	NA
$\Psi_{2,2}$	NA	0.6794***(0.1147)	0.0753(0.1510)	0.1745***(0.0544)	NA
$\Omega^{-1}_{v,1,1}$	NA	1.0391***(0.0556)	1.6717***(0.0569)	1.6727***(0.0549)	1.6993***(0.0548)
$\Omega^{-1}_{v,2,1}$	NA	0.1906(0.1799)	0.4147*(0.2169)	0.4099**(0.1948)	0.3222(0.2108)
$\Omega^{-1}_{v,2,2}$	NA	3.2280***(0.1685)	4.5593***(0.1527)	4.5856***(0.1176)	4.5999***(0.1509)
ν	NA	3.0951***(0.3974)	NA	NA	NA
C_1	NA	0.5187	0.6224	0.6450	0.2829
C_2	NA	NA	0.8011	0.8129	NA
C_2 to C_4 ADF	NA	All stationary	NA	NA	NA
C_3	NA	0.4638	NA	NA	NA
C_4	NA	0.2160	NA	NA	NA
Q_t ADF	NA	All stationary	NA	NA	NA
MDS $\epsilon_{1,t}$	NA	0.2127	0.6693	0.6695	0.9339
MDS $\epsilon_{2,t}$	NA	0.1559	0.9983	0.9763	0.7607
MDS $u_{1,t}$	NA	0.7120	NA	NA	NA
MDS $u_{2,t}$	NA	0.9384	NA	NA	NA
LL	NA	**−4.6943**	−4.8689	−4.8752	−4.8941
AIC	NA	**9.4460**	9.8000	9.8031	9.8313
BIC	NA	**9.5619**	9.9255	9.9093	9.9181
HQC	NA	**9.4918**	9.8496	9.8450	9.8656

For all models presented in this table, y_t is decomposed as

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}$$

and for each model the local level component is given by

$$\text{Local level for QVAR } \Psi \text{ diagonal:} \quad \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Psi_{1,1} & 0 \\ 0 & \Psi_{2,2} \end{bmatrix} \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix}$$

$$\text{Local level for VARMA } \Psi \text{ full:} \quad \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} \\ \Psi_{2,1} & \Psi_{2,2} \end{bmatrix} \begin{bmatrix} v_{1,t-1} \\ v_{2,t-1} \end{bmatrix}$$

$$\text{Local level for VAR:} \quad \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} v_{1,t-1} \\ v_{2,t-1} \end{bmatrix}$$

Not available (NA). We were not able to estimate QVAR 'Ψ full', as the ML estimator did not converge to an optimum. VARMA Ψ diagonal is obtained, by using the restrictions $\Psi_{1,2} = \Psi_{2,1} = 0$ for VARMA Ψ full. 'MDS' denotes the p-value of the martingale difference sequence test. Bold numbers indicate superior model performance. C_1 and C_2 are the maximum moduli of eigenvalues of Φ and Ψ, respectively. C_3 and C_4 are the maximum moduli of eigenvalues of $\hat{E}(X_t)$ and $\hat{E}(X_t \otimes X_t)$, respectively. We summarize ADF results for conditions 2 to 4 and matrix Q_t, by using 'All stationary'. Standard errors are in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Table 3.—Parameter Estimates and Model Diagnostics (Seasonal-QVAR; Basic Structural Model)

Seasonal-QVAR				Basic structural model	
c_1	0.1597***(0.0498)	$\gamma_{2,\mathrm{Jan}}$	−0.3705***(0.0681)	c_1	0.0862(0.2706)
c_2	0.1719(0.2009)	$\gamma_{2,\mathrm{Feb}}$	−0.2669***(0.0572)	c_2	0.1215(0.4161)
$\Phi_{1,1}$	0.3935**(0.1542)	$\gamma_{2,\mathrm{Mar}}$	−0.0590**(0.0254)	$\Phi_{1,1}$	−0.0947(0.0599)
$\Phi_{1,2}$	0.0417(0.0357)	$\gamma_{2,\mathrm{Apr}}$	0.1543***(0.0357)	$\Phi_{1,2}$	0.0316(0.0373)
$\Phi_{2,1}$	−0.5042(0.4401)	$\gamma_{2,\mathrm{May}}$	0.0530**(0.0235)	$\Phi_{2,1}$	−0.1555(0.3768)
$\Phi_{2,2}$	0.4989***(0.1023)	$\gamma_{2,\mathrm{Jun}}$	0.4799***(0.0839)	$\Phi_{2,2}$	0.5844***(0.1717)
$\Psi_{1,1}$	−1.0911(0.6735)	$\gamma_{2,\mathrm{Jul}}$	−0.1970***(0.0285)	$\Omega^{-1}_{v,1,1}$	0.2546(0.4974)
$\Psi_{1,2}$	−0.0104(0.0382)	$\gamma_{2,\mathrm{Aug}}$	−0.1387***(0.0380)	$\Omega^{-1}_{v,2,1}$	−3.1755***(0.5416)
$\Psi_{2,1}$	0.7832***(0.2552)	$\gamma_{2,\mathrm{Sep}}$	0.0235(0.0345)	$\Omega^{-1}_{v,2,2}$	0.0124***(0.0003)
$\Psi_{2,2}$	0.7953***(0.1192)	$\gamma_{2,\mathrm{Oct}}$	0.2720**(0.1068)	$\Omega^{-1}_{\eta,1,1}$	1.5925***(0.0916)
$\Omega^{-1}_{v,1,1}$	1.1981***(0.0646)	$\gamma_{2,\mathrm{Nov}}$	0.0595**(0.0270)	$\Omega^{-1}_{\eta,2,1}$	0.6450(0.9284)
$\Omega^{-1}_{v,2,1}$	−0.4530**(0.1765)	$\gamma_{2,\mathrm{Dec}}$	0.2695***(0.0628)	$\Omega^{-1}_{\eta,2,2}$	2.3133***(0.8054)
$\Omega^{-1}_{v,2,2}$	3.1445***(0.1466)	C_1	0.4662	$\sigma_{\xi,1}$	0.0413***(0.0138)
ν	3.4490***(0.4160)	C_2 to C_4 ADF	All stationary	$\sigma_{\xi,2}$	0.0697**(0.0322)
$\gamma_{1,\mathrm{Jan}}$	−0.3368(0.6920)	C_3	0.4365	C_1	0.5770
$\gamma_{1,\mathrm{Feb}}$	1.1408(0.8166)	C_4	0.2359	MDS $v_{1,t}$	0.9328
$\gamma_{1,\mathrm{Mar}}$	1.2451(0.7842)	Q_t ADF	All stationary	MDS $v_{2,t}$	0.6402
$\gamma_{1,\mathrm{Apr}}$	−0.5284(0.7023)	MDS $\epsilon_{1,t}$	0.6641	LL	−4.8033
$\gamma_{1,\mathrm{May}}$	0.9586(0.8113)	MDS $\epsilon_{2,t}$	0.3787	AIC	9.6735
$\gamma_{1,\mathrm{Jun}}$	0.6592(0.8052)	MDS $u_{1,t}$	0.3520	BIC	**9.8087**
$\gamma_{1,\mathrm{Jul}}$	0.5037(0.9552)	MDS $u_{2,t}$	0.1269	HQC	**9.7269**
$\gamma_{1,\mathrm{Aug}}$	0.8943(0.6866)	LL	**-4.7442**		
$\gamma_{1,\mathrm{Sep}}$	0.3072(0.6728)	AIC	**9.6702**		
$\gamma_{1,\mathrm{Oct}}$	2.8221***(1.0550)	BIC	10.0371		
$\gamma_{1,\mathrm{Nov}}$	0.6534(0.6349)	HQC	9.8153		
$\gamma_{1,\mathrm{Dec}}$	0.6847(0.8062)				

For all models presented in this table, y_t is decomposed as

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} + \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}$$

and for each model the local level component is given by

Local level for Seasonal-QVAR: $$\begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} \\ \Psi_{2,1} & \Psi_{2,2} \end{bmatrix} \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix}$$

Local level for the basic structural model: $$\begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \end{bmatrix} + \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix}$$

Seasonality for Seasonal-QVAR: $$\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} D_t'\rho_{1,t} \\ D_t'\rho_{2,t} \end{bmatrix} = \begin{bmatrix} D_t'(\rho_{1,t-1} + \gamma_{1,t}u_{1,t-1}) \\ D_t'(\rho_{2,t-1} + \gamma_{2,t}u_{2,t-1}) \end{bmatrix}$$

Seasonality for the basic structural model: $$\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} D_t'\rho_{1,t} \\ D_t'\rho_{2,t} \end{bmatrix} = \begin{bmatrix} D_t'(\rho_{1,t-1} + \xi_{1,t}) \\ D_t'(\rho_{2,t-1} + \xi_{2,t}) \end{bmatrix}$$

'MDS' denotes the p-value of the martingale difference sequence test. Bold numbers indicate superior model performance. C_1 is the maximum modulus of eigenvalues of Φ. C_3 and C_4 are the maximum moduli of eigenvalues of $\hat{E}(X_t)$ and $\hat{E}(X_t \otimes X_t)$, respectively. We summarize ADF results for conditions 2 to 4 and matrix Q_t, by using 'All stationary'. Standard errors are in parentheses. ** and *** indicate significance at the 5% and 1% levels, respectively.

Table 4.—Parameter Estimates and Model Diagnostics (Seasonal-VARMA; Seasonal-VAR)

Seasonal-VARMA				Seasonal-VAR			
c_1	$-0.0122(0.0473)$	$\gamma_{2,\text{Jan}}$	$-0.0584^{***}(0.0094)$	c_1	$-0.0794(0.0820)$	$\gamma_{2,\text{Jan}}$	$-0.0457^{***}(0.0131)$
c_2	$0.2549^{***}(0.0754)$	$\gamma_{2,\text{Feb}}$	$-0.0902^{***}(0.0185)$	c_2	$0.3662^{***}(0.1043)$	$\gamma_{2,\text{Feb}}$	$-0.0709^{***}(0.0133)$
$\Phi_{1,1}$	$0.6248^{***}(0.1031)$	$\gamma_{2,\text{Mar}}$	$-0.0056(0.0208)$	$\Phi_{1,1}$	$0.0437(0.1025)$	$\gamma_{2,\text{Mar}}$	$0.0024(0.0154)$
$\Phi_{1,2}$	$0.0484(0.0335)$	$\gamma_{2,\text{Apr}}$	$0.0431^{***}(0.0112)$	$\Phi_{1,2}$	$-0.0002(0.0208)$	$\gamma_{2,\text{Apr}}$	$0.0503^{***}(0.0281)$
$\Phi_{2,1}$	$-0.7541^{***}(0.2900)$	$\gamma_{2,\text{May}}$	$0.0356(0.0359)$	$\Phi_{2,1}$	$0.1122(0.0853)$	$\gamma_{2,\text{May}}$	$0.0245(0.0099)$
$\Phi_{2,2}$	$0.3865^{***}(0.0647)$	$\gamma_{2,\text{Jun}}$	$0.0839^{***}(0.0094)$	$\Phi_{2,2}$	$0.3073^{***}(0.0415)$	$\gamma_{2,\text{Jun}}$	$0.0945^{***}(0.0073)$
$\Psi_{1,1}$	$-0.2706^{***}(0.0940)$	$\gamma_{2,\text{Jul}}$	$-0.0695^{***}(0.0087)$	$\Psi_{1,1}$	NA	$\gamma_{2,\text{Jul}}$	$-0.0622^{***}(0.0078)$
$\Psi_{1,2}$	$0.0003(0.0171)$	$\gamma_{2,\text{Aug}}$	$-0.0318^{***}(0.0085)$	$\Psi_{1,2}$	NA	$\gamma_{2,\text{Aug}}$	$-0.0300^{***}(0.0194)$
$\Psi_{2,1}$	$0.1361^{**}(0.0655)$	$\gamma_{2,\text{Sep}}$	$0.0319^{*}(0.0191)$	$\Psi_{2,1}$	NA	$\gamma_{2,\text{Sep}}$	$0.0653^{***}(0.0080)$
$\Psi_{2,2}$	$0.3169^{***}(0.0423)$	$\gamma_{2,\text{Oct}}$	$-0.0039(0.0145)$	$\Psi_{2,2}$	NA	$\gamma_{2,\text{Oct}}$	$-0.0053(0.0250)$
$\Omega^{-1}_{v,1,1}$	$1.8231^{***}(0.0424)$	$\gamma_{2,\text{Nov}}$	$-0.0242(0.0257)$	$\Omega^{-1}_{v,1,1}$	$1.8581^{***}(0.0576)$	$\gamma_{2,\text{Nov}}$	$-0.0247(0.0315)$
$\Omega^{-1}_{v,2,1}$	$-0.3318^{***}(0.0537)$	$\gamma_{2,\text{Dec}}$	$0.1632^{***}(0.0296)$	$\Omega^{-1}_{v,2,1}$	$-0.4779^{***}(0.0706)$	$\gamma_{2,\text{Dec}}$	$0.1513^{***}(0.0000)$
$\Omega^{-1}_{v,2,2}$	$4.1728^{***}(0.1040)$	C_1	0.5272	$\Omega^{-1}_{v,2,2}$	$4.2325^{***}(0.1307)$	C_1	0.3072
$\gamma_{1,\text{Jan}}$	$-0.8857^{***}(0.0672)$	C_2	0.8398	$\gamma_{1,\text{Jan}}$	$-1.1382^{***}(0.2131)$	C_2	NA
$\gamma_{1,\text{Feb}}$	$0.1523(0.1702)$	MDS $\epsilon_{1,t}$	0.5582	$\gamma_{1,\text{Feb}}$	$-0.1301(0.2093)$	MDS $\epsilon_{1,t}$	0.1692
$\gamma_{1,\text{Mar}}$	$0.3098^{**}(0.1216)$	MDS $\epsilon_{2,t}$	0.2022	$\gamma_{1,\text{Mar}}$	$-0.0357(0.1643)$	MDS $\epsilon_{2,t}$	0.3398
$\gamma_{1,\text{Apr}}$	$0.0909(0.0777)$	LL	**−4.8670**	$\gamma_{1,\text{Apr}}$	$-0.1655(0.1305)$	LL	−4.9002
$\gamma_{1,\text{May}}$	$0.1898(0.1223)$	AIC	**9.9111**	$\gamma_{1,\text{May}}$	$-0.0176(0.2365)$	AIC	9.9583
$\gamma_{1,\text{Jun}}$	$0.1578^{**}(0.0732)$	BIC	**10.2683**	$\gamma_{1,\text{Jun}}$	$-0.1143(0.1367)$	BIC	10.2769
$\gamma_{1,\text{Jul}}$	$0.9664^{***}(0.1335)$	HQC	**10.0523**	$\gamma_{1,\text{Jul}}$	$0.7054^{***}(0.1601)$	HQC	10.0842
$\gamma_{1,\text{Aug}}$	$0.2705^{***}(0.0999)$			$\gamma_{1,\text{Aug}}$	$-0.0361(0.1404)$		
$\gamma_{1,\text{Sep}}$	$0.0219(0.1544)$			$\gamma_{1,\text{Sep}}$	$-0.2765^{**}(0.1372)$		
$\gamma_{1,\text{Oct}}$	$0.6388^{***}(0.0905)$			$\gamma_{1,\text{Oct}}$	$0.3391^{*}(0.1930)$		
$\gamma_{1,\text{Nov}}$	$0.0520(0.1538)$			$\gamma_{1,\text{Nov}}$	$-0.3637^{***}(0.1405)$		
$\gamma_{1,\text{Dec}}$	$-0.1005(0.1893)$			$\gamma_{1,\text{Dec}}$	$-0.7315^{***}(0.1701)$		

For all models presented in this table, y_t is decomposed as

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} + \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}$$

and for each model the local level component is given by

Local level for Seasonal-VARMA: $$\begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} \\ \Psi_{2,1} & \Psi_{2,2} \end{bmatrix} \begin{bmatrix} v_{1,t-1} \\ v_{2,t-1} \end{bmatrix}$$

Local level for Seasonal-VAR: $$\begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} \mu_{1,t-1} \\ \mu_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} v_{1,t-1} \\ v_{2,t-1} \end{bmatrix}$$

Seasonality for both models: $$\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} D_t'\rho_{1,t} \\ D_t'\rho_{2,t} \end{bmatrix} = \begin{bmatrix} D_t'(\rho_{1,t-1} + \gamma_{1,t}v_{1,t-1}) \\ D_t'(\rho_{2,t-1} + \gamma_{2,t}v_{2,t-1}) \end{bmatrix}$$

An identical representation of Seasonal-VARMA and Seasonal-VAR is presented in equation (13). ‘MDS’ denotes the p-value of the martingale difference sequence test. Bold numbers indicate superior model performance. C_1 and C_2 are the maximum moduli of eigenvalues of Φ and $\Psi - \Phi$, respectively. Standard errors are in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

FIGURE 1.—IMPULSE RESPONSE FUNCTION (QVAR; VARMA; VAR)

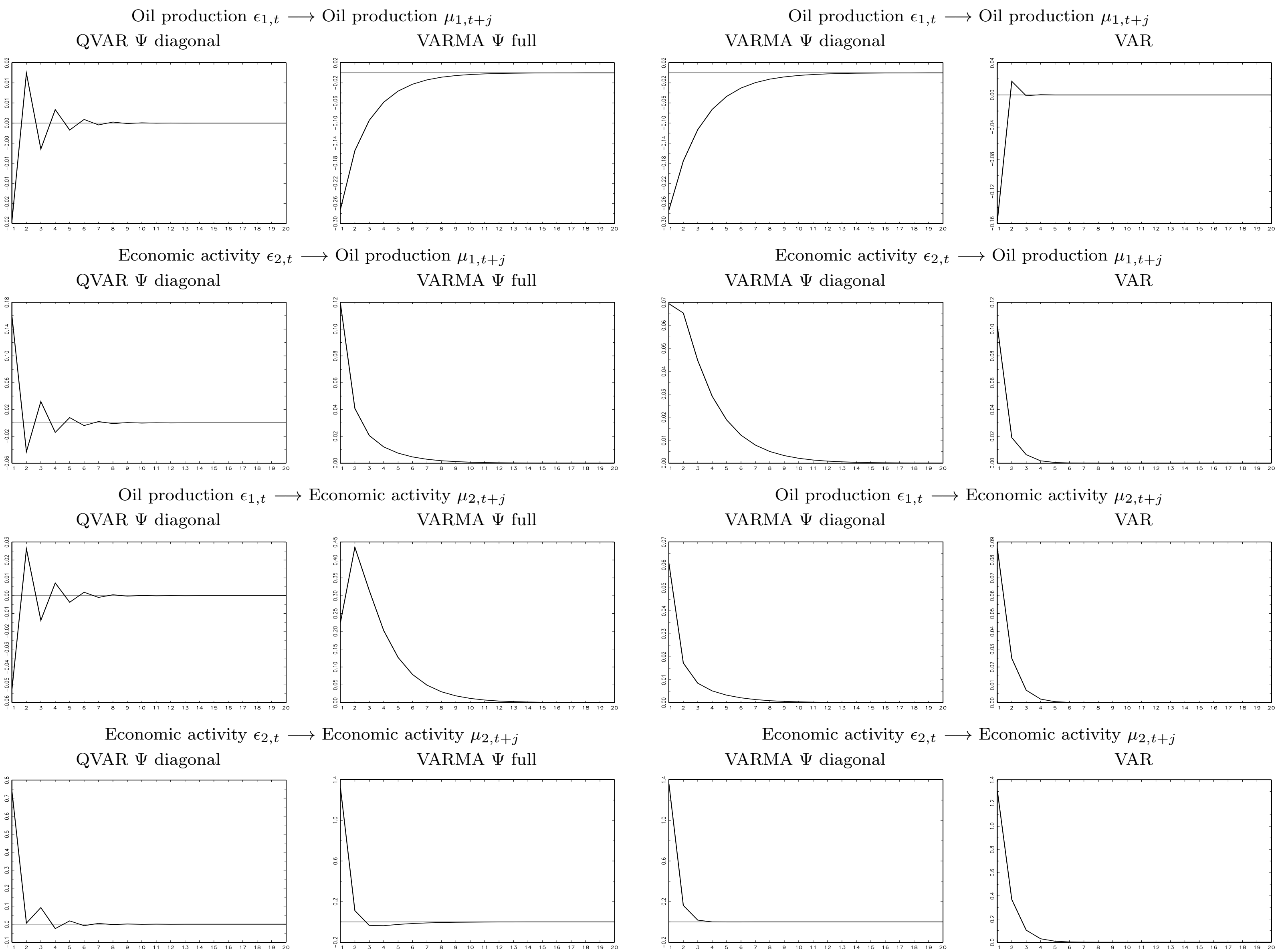

Figure 2.—Time Series Components of Seasonal-QVAR

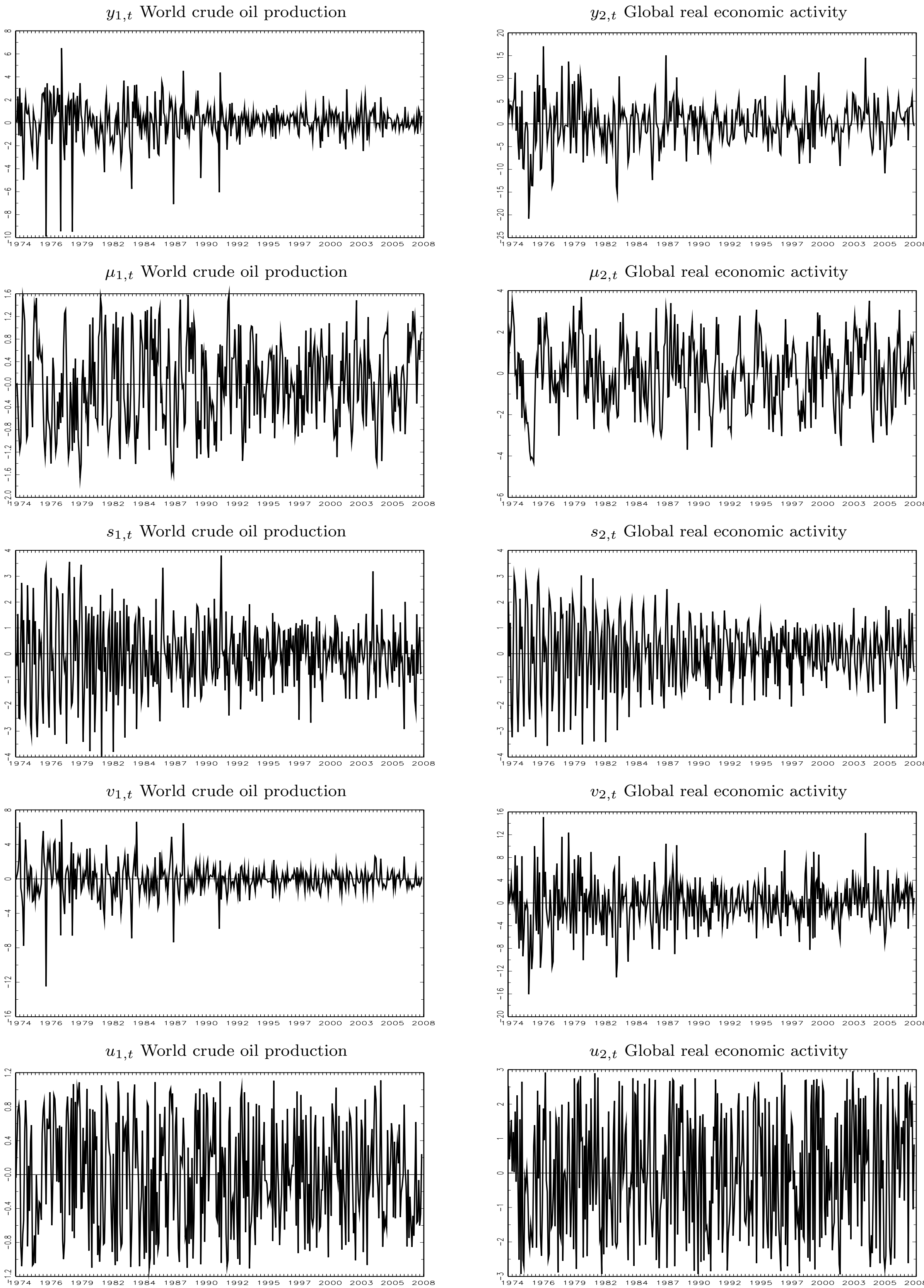

Figure 3.—Time Series Components of the Basic Structural Model

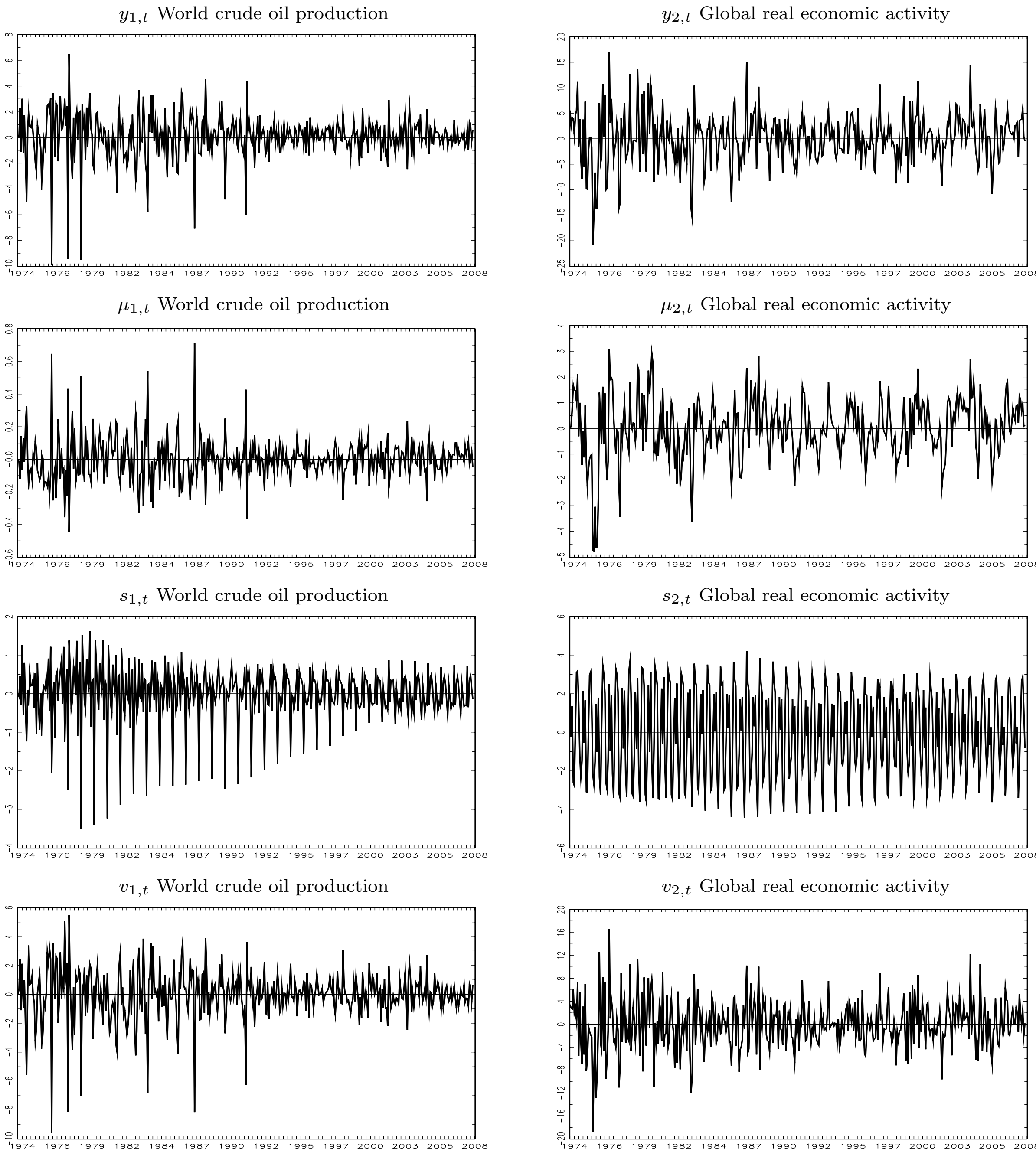

Figure 4.—Time Series Components of Seasonal-VARMA

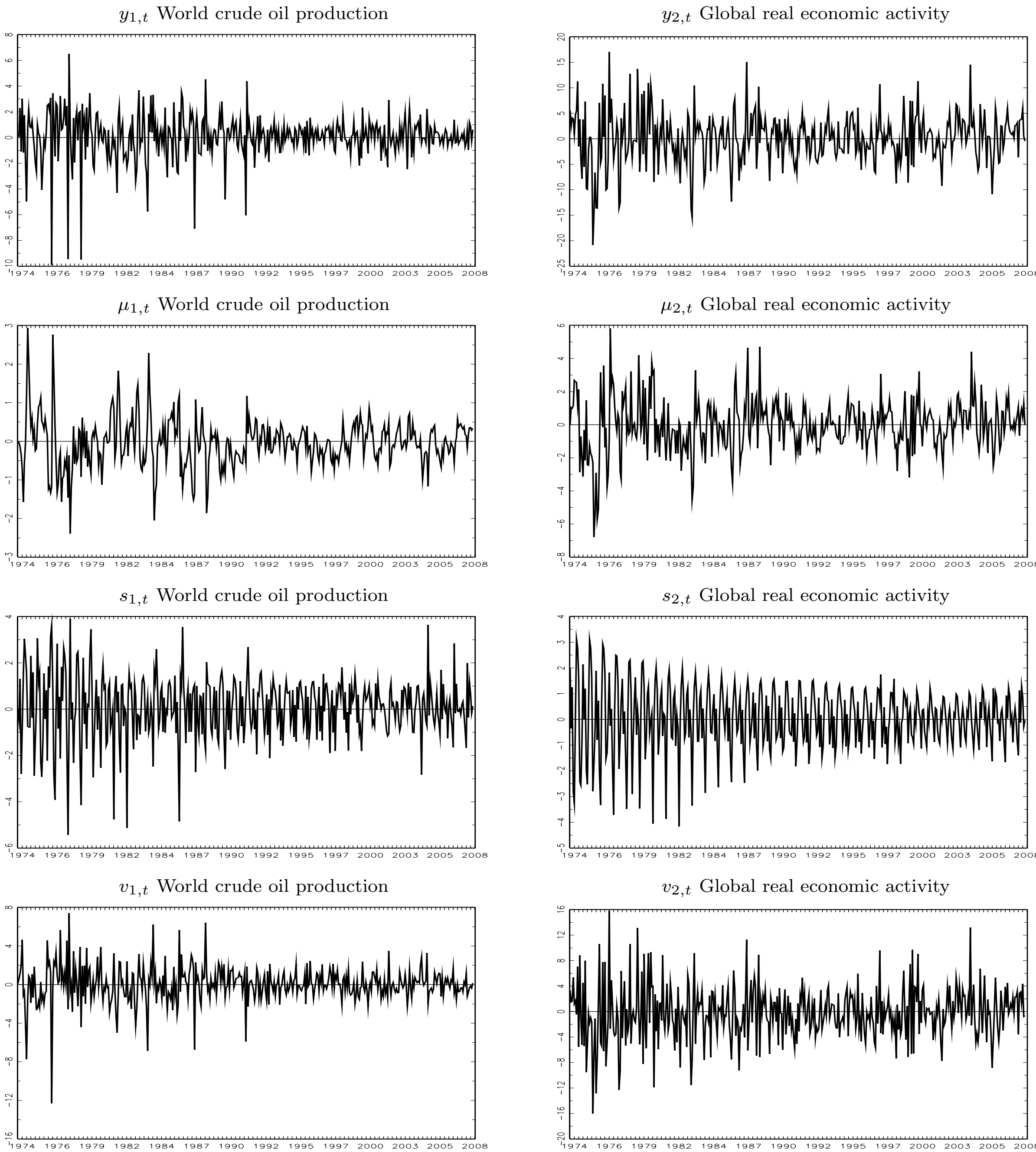

Figure 5.—Time Series Components of Seasonal-VAR

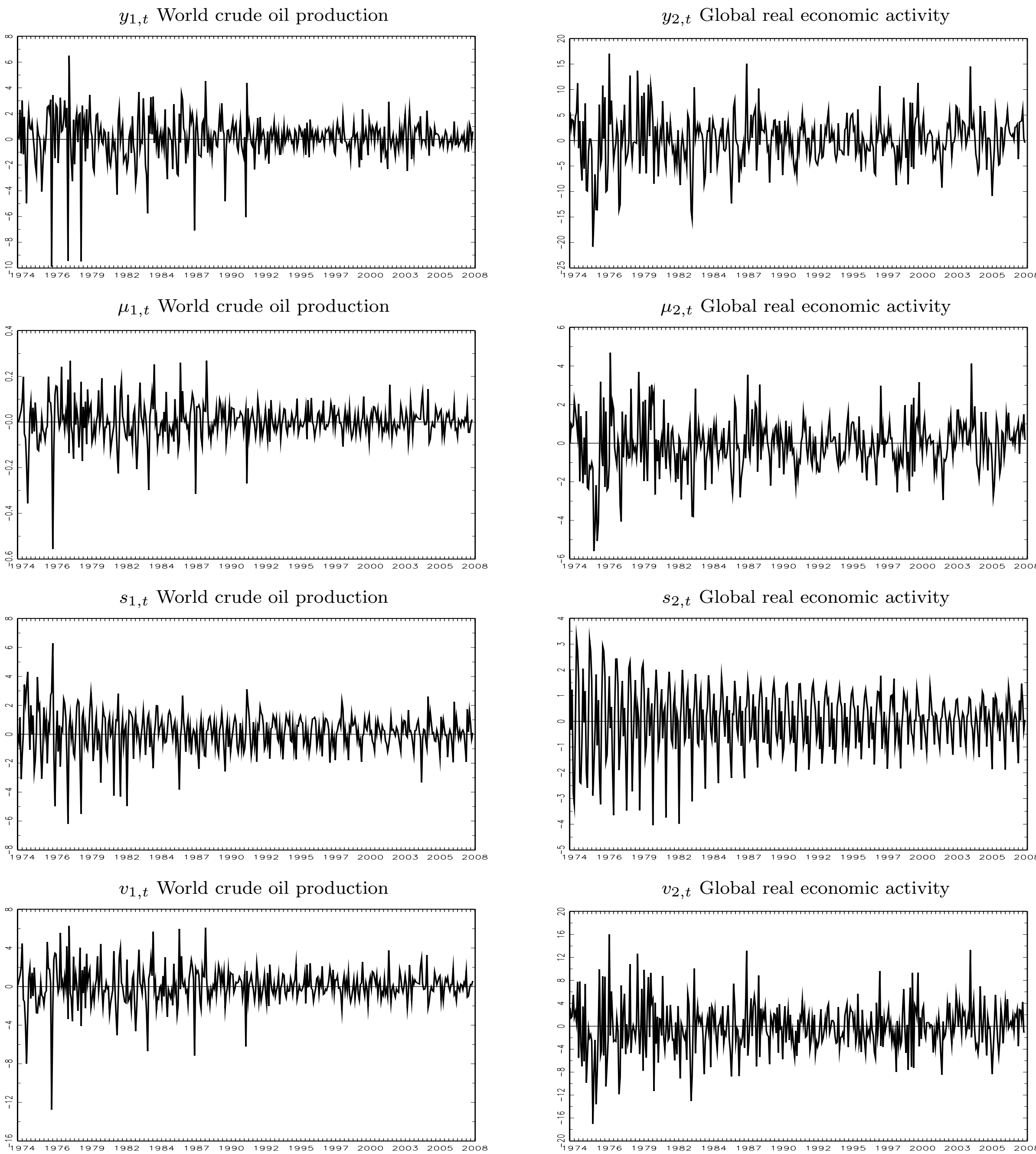

Figure 6.—Impulse Response Function (Seasonal-QVAR; Basic Structural Model; Seasonal-VARMA; Seasonal-VAR)

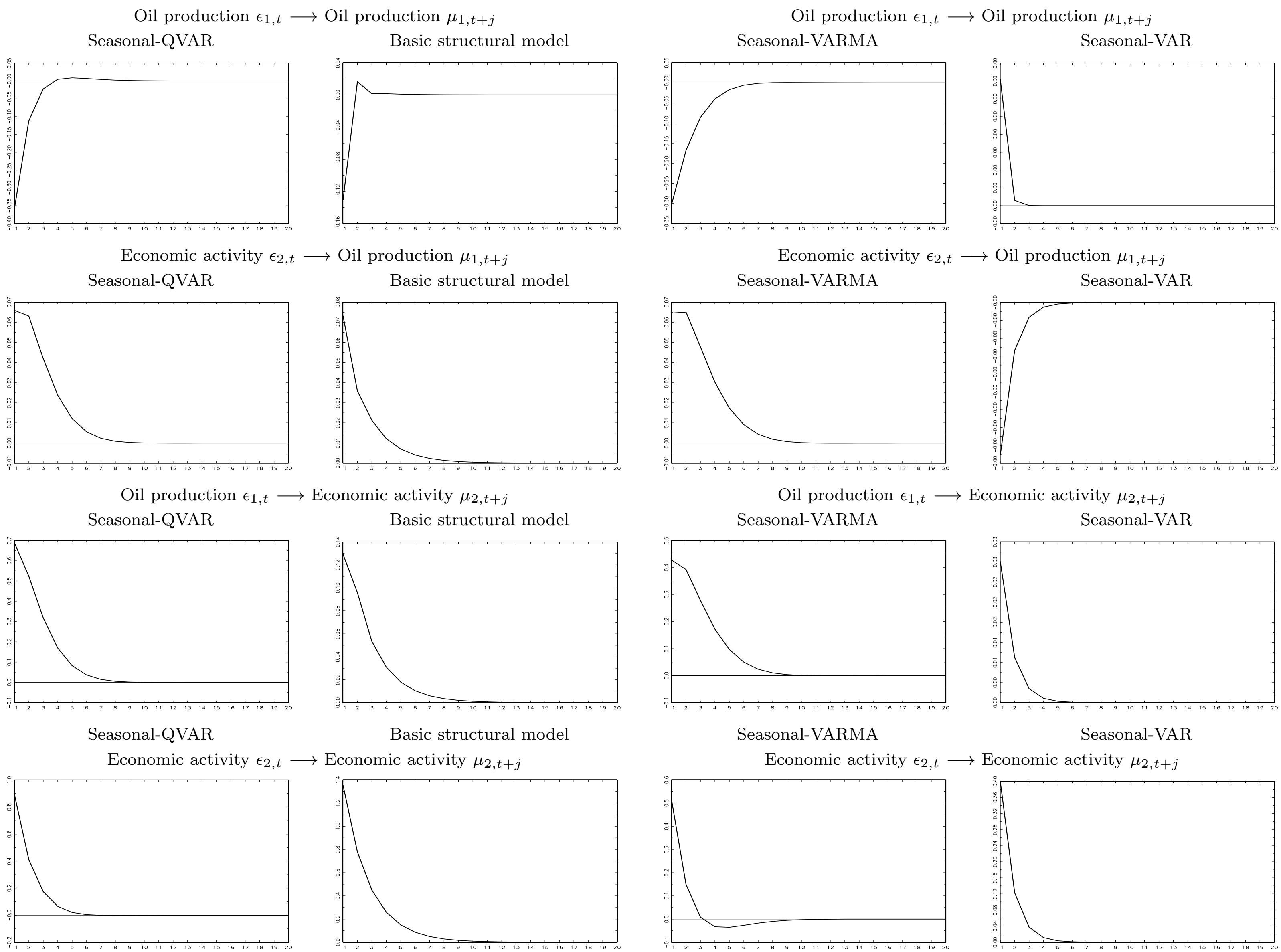

Figure 7.—Robustness to Extreme Values in the Noise (Seasonal-QVAR; Seasonal-VARMA)